On the c^2 term in the holographic formula for dark energy

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Abstract

It is argued that the c^2 term that appears in the conventional formula for holographic dark energy should not be assumed constant in general. Notwithstanding, there is at least an exception, namely, when the Ricci scale is chosen as the infrared cutoff length.

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I. INTRODUCTION

Cosmological models based on holographic dark energy rest on the rather reasonable assumption that the entropy of every bounded region of the Universe, of size L, should not exceed the entropy of a Schwarzschild black hole of the same size. Mathematically,

$$L^3 \Lambda^3 \leq S_{BH} \simeq L^2 M_{Pl}^2 \qquad (M_{Pl}^2 = (8\pi G)^{-1}) \,, \tag{1}$$

where Λ stands for the ultraviolet cutoff while the infrared cutoff is set by L.

However, an effective field theory that saturates the above inequality necessarily includes states such that the Schwarzschild radius exceeds L [1]. It is therefore natural to replace the above bound by another not allowing such states,

$$L^3 \Lambda^4 \leq M_{Pl}^2 L \,. \tag{2}$$

This bound guarantees that the energy $L^3\Lambda^4$ in a region of the size L does not exceed the energy of a black hole of the same size [2]. By saturating the inequality (2) and identifying Λ^4 with the density of holographic dark energy, ρ_x, it follows that [1, 2]

$$\rho_x = \frac{3c^2}{8\pi G \, L^2} \,, \tag{3}$$

where the factor 3 was introduced for convenience and c^2 is a dimensionless quantity that collects the uncertainties of the theory (such as the number particle species and so on). On the other hand, a further interesting feature of holography lies in its close connection with the spacetime foam, as unveiled by Ng [3, 4]. Additional motivations for holographic dark energy can be found in Section 3 of [5].

Very often, for the sake of simplicity, the c^2 parameter is assumed constant. However, one should bear in mind that it is more general to consider it a slowly varying function of time, $c^2(t)$, as in [6–8]. By slowly varying we mean that $(c^2)^{\cdot}/c^2$ is upper bounded by the Hubble expansion rate, H, i.e.,

$$\frac{(c^2(t))^{\cdot}}{c^2(t)} \lesssim H \,. \tag{4}$$

Note that this condition must be fulfilled at all times; otherwise the dark energy density would not even approximately be proportional to L^{-2}, something at the core of holography.

The target of this work is to argue that, rather generally, it is not consistent to consider c^2 constant. Obviously, c^2 will depend on the infrared length, L assumed by the model. We

shall consider three different lengths: the Hubble length, H^{-1}, the particle horizon length (defined below by Eq. (10)), and Ricci's length, $L = (\dot{H} + 2H^2)^{-1/2}$. We shall not include in our discussion the widely used length defined by the radius of the future event horizon because the corresponding models suffer from a severe circularity problem. Namely, the event horizon is needed to have acceleration and vice versa.

As for dark energy, we shall consider first the cosmological constant and then dark energy with arbitrary equation of state. In all the cases, we shall limit ourselves to spatially flat universes described by the Friedmann-Robertson-Walker (FRW) metric.

II. DARK ENERGY GIVEN BY THE COSMOLOGICAL CONSTANT

Nowadays the spatially flat ΛCDM model is, from the observational point of view, the leading cosmological model in the market -see, e.g. [9–12]. It assumes that the energy budget is dominated by the constant dark energy density of the quantum vacuum, ρ_Λ, and a dust contribution that redshifts with expansion quickly while radiation and other forms of energy are negligible at present, which also redshift with expansion. This implies that $\rho_\Lambda \to \rho_{total}$ as $t \to \infty$.

Before going any further one may ask, in the first place, why ρ_Λ should be holographic at all. Recall that the entropy of the quantum vacuum is a constant that may be fixed to zero. We have no clear answer to this point. However, if dark energy with equation of state $w = -1 + \epsilon$ (with $|\epsilon| \ll 1$) is holographic, then ρ_Λ should also be holographic, on account of continuity, when $\epsilon \to 0$. Thus, we shall tentatively assume that ρ_Λ obeys Eq. (3).

To motivate that c^2 must vary with time we take first the infrared cutoff provided by the Hubble length, $L = H^{-1}$. Barring the presence of fields that violate the dominant energy condition, we can replace ρ_{total} by ρ_Λ at very late times, whence in virtue of the first Friedmann equation, we can write

$$\rho_\Lambda = \frac{3}{8\pi G}\, c^2(t \to \infty)\, H^2(t \to \infty) = \frac{3}{8\pi G}\, H^2(t \to \infty) \qquad \Rightarrow \qquad c^2(t \to \infty) := c^2_\infty = 1\,. \tag{5}$$

Obviously, c^2 cannot be unity at earlier times because it would not leave room for any other forms of energy whatsoever (matter, radiation, ...). On the other hand it is obvious that c^2 approaches unity from below.

Another point to consider is whether c^2 varies sufficiently slow at all times, i.e., whether the condition (4) is met at all epochs. To elucidate this we recall that $\rho_\Lambda = $ constant. Then,

$$c^2(t)\, H^2(t) = c^2_\infty\, H^2_\infty = \text{constant} \qquad \Rightarrow \qquad [c^2\, H^2]^{\cdot} = 0\,. \tag{6}$$

To make matters easier, assume that the Universe is dominated solely by pressureless matter (subscript m) and ρ_Λ. Using the second Friedmann equation, $\dot{H} = -4\pi G \rho_m$, and $\Omega_m := 8\pi G\rho_m/(3H^2)$ (bear in mind that Ω_m varies with time), we get

$$\frac{(c^2)^{\cdot}}{c^2} = 3\,\Omega_m\, H\,. \tag{7}$$

At early times, $\Omega_m \simeq 1$ and condition (4) is violated. We conclude by saying that if $L = H^{-1}$, then the energy density of the quantum vacuum is not holographic at early times; it might be thought as holographic only when $\Omega_m \lesssim 1/3$.

Assume now that the infrared cutoff is given by Ricci's length, $L = (\dot{H} + 2H^2)^{-1/2}$, as in [13]. Since ρ_Λ is to completely dominate over non-relativistic matter when $t \rightarrow \infty$ it follows that

$$\rho_\Lambda = \frac{3H^2_\infty}{8\pi G} = \text{constant} = \frac{3c^2}{8\pi G}\,(\dot{H}(t)\, +\, 2H^2(t))\,, \tag{8}$$

but $\dot{H}(t \rightarrow \infty)$ vanishes because $\dot{H} = -4\pi G\rho_m$. As a consequence, $c^2_\infty = 1/2$. This value is attained asymptotically from below.

Thus, as expected, the asymptotic value of c^2 depends on the infrared cutoff.

To elucidate whether condition (4) is fulfilled, we proceed as before. We start from $[c^2\,(\dot{H}\, +\, 2H^2)]^{\cdot} = 0$ and use $\ddot{H} = -4\pi G\dot{\rho}_m = 12\pi G H \rho_m$ alongside the expressions for $\dot{H}$ and Ω_m of above to obtain

$$\frac{(c^2)^{\cdot}}{c^2} = \frac{3\,\Omega_m}{4\, -\, 3\Omega_m} H\,. \tag{9}$$

In consequence, when $2/3 \lesssim \Omega_m \lesssim 1$ (i.e., early times) one has $H \lesssim (c^2)^{\cdot}/c^2$. So, in this case as well ρ_Λ fails to be holographic.

We next take the particle horizon,

$$L = a(t)\int_0^t \frac{dt'}{a(t')} = a\int_0^a \frac{da'}{a'^2 H(a')}, \tag{10}$$

where a is the scale factor of the FRW metric, as the infrared cutoff. For ρ_Λ to be consistent with (3), $c(t)$ must vary as L.

For a spatially flat FRW universe dominated by dust and a cosmological constant, the first Friedmann's equation reduces to

$$H^2 = \frac{8\pi G \rho_\Lambda}{3} \left[\frac{r_0}{a^3} + 1 \right] , \tag{11}$$

where r_0 denotes the present value of the ratio between the densities of matter and vacuum energy, $r \equiv \rho_m/\rho_\Lambda$. In this case L can be expressed in terms of a hypergeometric function, and we obtain

$$c(a) = \sqrt{\frac{8\pi G \rho_\Lambda}{3}} L = 2\sqrt{\frac{a^3}{r_0}} \; {}_2F_1 \left(1/6, 1/2, 7/6; -a^3/r_0 \right) . \tag{12}$$

Next we check whether for such a choice of the infrared cutoff c^2 varies slowly enough. Condition (4) leads to

$$\frac{(c^2)^{\cdot}}{c^2} = \frac{(L^2)^{\cdot}}{L^2} = 2 \left[H + \frac{1}{a^2 H \int_0^a \frac{da'}{a'^2 H(a')}} \right] \lesssim H . \tag{13}$$

That is to say,

$$H + \frac{2}{\sqrt{\frac{a^5}{r_0}} H \; {}_2F_1 \left(1/6, 1/2, 7/6; -a^3/r_0 \right)} < 0 . \tag{14}$$

Obviously the inequality fails since all the left hand side terms are non-negative. In particular, the hypergeometric function, ${}_2F_1 \left(1/6, 1/2, 7/6; -a^3/r_0 \right)$, varies monotonously from zero at $a = 0$ to $2\, r_0^{1/3} \Gamma(1/3) \Gamma(7/6)/\sqrt{\pi}$ when $a \to \infty$.

III. DARK ENERGY WITH GENERIC EQUATION OF STATE

Consider the Universe dominated by pressureless matter and dark energy (subindexes x and m, respectively) and assume that they interact with each other gravitationally only. Then,

$$\rho_m = \rho_{m0} \left(1 + z \right)^3 , \quad \text{and} \quad \rho_x = \rho_{x0} \, \exp \left[3 \int_0^z \left(\frac{1 + w(z')}{1 + z'} \right) dz' \right] . \tag{15}$$

(We note in passing that, in general, the equation of state parameter of dark energy, w, is not constant in this context). On the other hand, if the dark energy is holographic with the infrared length given by the Hubble radius, $L = H^{-1}$, then

$$\rho_x = \frac{3c^2}{8\pi G} H^2 . \tag{16}$$

Recalling the first Friedmann's equation, $3H^2 = 8\pi G(\rho_m + \rho_x)$, we can write

$$\frac{3H^2}{8\pi G}(1 - c^2) = \rho_{m0}\,(1+z)^3 \tag{17}$$

as well as

$$\frac{3H^2}{8\pi G} = \rho_{m0}\,(1+z)^3 + \rho_{x0}\,\exp\left[3\int_0^z \left(\frac{1+w(z')}{1+z'}\right) dz'\right]. \tag{18}$$

After combining last two equations and simplifying, we get

$$r = \frac{1 - c^2}{c^2} = r_0\,(1+z)^3\,\exp\left[-3\int_0^z \left(\frac{1+w(z')}{1+z'}\right) dz'\right]. \tag{19}$$

That is to say, as the second equality tells us, the c^2 parameter cannot but vary with expansion. Moreover, as $z \to -1$ (i.e., $t \to \infty$), $c^2 \to 1$ from below and $r \to 0$.

We next study whether c^2 evolves slowly enough. To this end we use the conservation equation $\dot{\rho_x} + 3H(1+w)\rho_x = 0$ alongside Eq. (16) to obtain

$$\frac{(c^2)^{\cdot}}{c^2}H + 2\dot{H} = -3(1+w)H^2\,.$$

With the help of the first Friedmann equation (above) and the second one, $\dot{H} = -4\pi G\left[\rho_m + (1+w)\rho_x\right]$, it follows that $\left((c^2)^{\cdot}/c^2\right)H = -8\pi G w \rho_m$. Dividing by H^2 yields

$$\frac{(c^2)^{\cdot}}{c^2} = -3Hw\,\Omega_m\,. \tag{20}$$

Therefore, since for dark energy $w < 0$ we see that c^2 augments with expansion. An additional consequence is that c^2 will vary sufficiently slow for $3\,|w|\,\Omega_m \lesssim 1$ only.

Let us continue by considering a holographic dark energy, whose evolution is formally given by (15), but choosing as infrared cutoff the Ricci's length, $L = (\dot{H} + 2H^2)^{-1/2}$. Thus,

$$\frac{c^2}{L^2} = \frac{8\pi G}{3}\,\rho_{x0}\,\exp\left[3\int_0^z \left(\frac{1+w(z')}{1+z'}\right) dz'\right], \tag{21}$$

i.e.,

$$c^2 = \frac{8\pi G}{3}\,\frac{\rho_{x0}}{\dot{H} + 2H^2}\,\exp\left[3\int_0^t (1+w(t'))\,H(t')dt'\right]. \tag{22}$$

From the field equations, and after some algebra, one can obtain $c^2\,(1 + r - 3w) = 2$. This implies that

$$\frac{(c^2)^{\cdot}}{c^2} = \frac{\frac{\dot{w}}{w} - Hr}{\frac{1}{3w\Omega_x} - 1}. \tag{23}$$

Consequently, condition (4) amounts to

$$\frac{\dot{w}}{w} \lesssim H \left[r - 1 + \frac{1}{3w\Omega_x} \right] . \tag{24}$$

In this case a constant c^2 is admissible provided the dark energy density obeys [13]

$$\rho_x = \frac{\rho_{m0} c^2}{2 - c^2} \left[a^{-3} + k a^{-4+2/c^2} \right] , \tag{25}$$

where k is an integration constant.

To check whether the bound (4) is fulfilled some expression for w is needed. We choose the widely used Chevallier-Polarski-Linder parametrization [16, 17],

$$w(z) = w_0 + w_1 \frac{z}{1+z} . \tag{26}$$

The constant parameters w_0 and w_1 are observationally constrained by supernovae, cosmic background radiation, and large scale structure data, see e.g. [18],

$$w_0 = -0.90^{+0.11}_{-0.11} , \qquad w_1 = -0.24^{+0.56}_{-0.55} . \tag{27}$$

As inspection of Fig. 1 reveals, the holographic constraint is fulfilled only at 2σ confidence level.

Finally, we repeat the analysis but now we choose the particle horizon, defined in Eq. (10), as infrared cutoff. In this case,

$$c^2(a) = a^2 f^2(a) \left[\int_0^a \frac{da'}{\sqrt{r_0 a' + a'^4 f^2(a')}} \right]^2 , \tag{28}$$

where

$$f(a) = \exp \left[-\frac{3}{2} \int \frac{1 + w(a')}{a'} da' \right] . \tag{29}$$

While Eq. (28) does not completely exclude the case of a constant c^2, this possibility looks rather slim as $w(a)$ should take a very contrived expression.

To ascertain whether the bound (4) can be satisfied we write the derivative of c^2 from the conservation equation

$$\frac{(c^2)^{\cdot}}{c^2} = 2H \left[1 + \frac{\sqrt{\Omega_x}}{c} \right] - 3H(1+w) \tag{30}$$

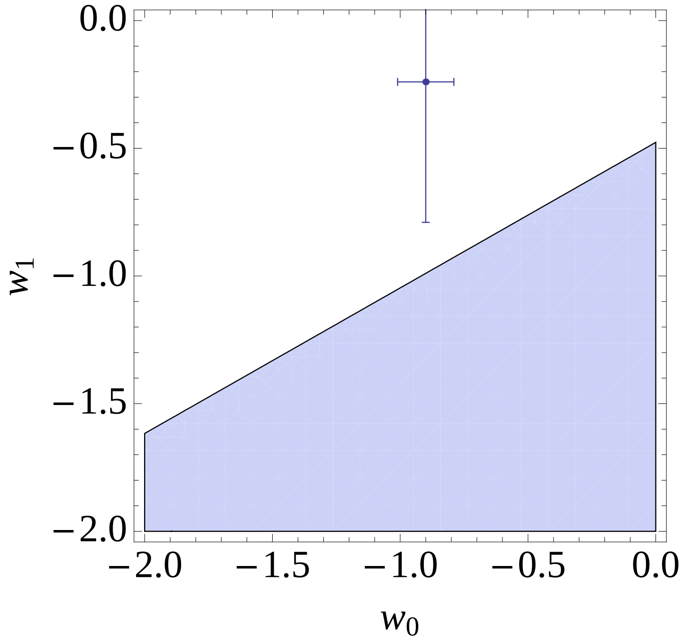

Figure 1. The dark zone in the plane (w_0, w_1) corresponds to the parameter space region in which the inequality (24), for $z = 0$, is satisfied. Also indicated is the best fit value with the 1σ uncertainties for the pair (w_0, w_1) as reported in [18], -see Eqs. (27).

and use $c/\sqrt{\Omega_x} = LH$ to get

$$\frac{(c^2)^{\cdot}}{c^2} = H\left\{2\left(1 + \frac{1}{\sqrt{r_0 a^{-1} + a^2 f^2(a)}\int_0^a \frac{da'}{a\sqrt{r_0 a'^{-1} + a'^2 f^2(a')}}}\right) - 3(1+w)\right\}. \tag{31}$$

So, the term in the curly parenthesis should be lower than unity. Even being rather conservative, inspection readily reveals that for this to occur w should be larger than $-2/3$ at all times, which is widely excluded by observation.

IV. CONCLUDING REMARKS

Altogether, except for the particular case of dark energy with the Ricci's length as infrared cutoff, in all the instances examined, the c^2 term in the widely used holographic expression (3) for dark energy should not be assumed constant. Generally, it varies faster than the Hubble rate at least for some long periods of expansion. Even in the said particular case, using the Chevallier-Polarski-Linder parametrization [16, 17], the bound (4) is satisfied at just at 2σ confidence level.

Clearly the bound (4) looks reasonable but, notwithstanding, it is debatable. On the one hand if c^2 varied much more faster than the scale factor, then holography would break down (i.e., the entropy of a region of size L would not longer be proportional to L^2). On the other hand, we do not know of any definitive argument to adamantly enforce it. One may contend that H could be replaced by $n\,H$ on the right hand side of (4), with n a positive constant of order unity. However, the precise value n should take it is rather a matter of choice. Therefore, for the sake of definiteness, we believe it is better simply to keep $n = 1$.

We have not considered possible non-gravitational interactions in the dark sector, very often invoked to alleviate the coincidence problem -see eg. [10, 19, 20] and references therein. It remains to be seen which specific interactions are consistent with $c^2 =$ constant or, more generally, with the bound (4). We defer this study to a future work.

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